Minmax Methods for Geodesics and Minimal Surfaces

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Lecture 3 : A Viscosity Approach to the Minmax Theory of Geodesics and Minimal Surfaces.

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Proposition $(\mathcal{M}, \|\cdot\|)$ *defines a* **complete Finsler manifold**. E^σ *is* C^1 *on* $\mathcal{M}$

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Do we have $\beta_0 = L(\vec{\gamma}_0)$ and $\vec{\gamma}_0$ is a geodesic ?

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$$E^{\sigma_j}(\vec{\gamma}_{\sigma_j}) = \beta_{\sigma_j} \quad , \quad DE^{\sigma_j}_{\vec{\gamma}_{\sigma_j}} = 0$$

and

$$\sigma_j^2 \int_{S^1} \kappa^2_{\vec{\gamma}_{\sigma_j}}\ dl_{\vec{\gamma}_{\sigma_j}} = o\left(\frac{1}{\log\left(\frac{1}{\sigma_j}\right)}\right) \quad .$$

then $\exists\ \sigma_{j'} \to 0$

$$\frac{d\vec{\gamma}_{\sigma_j}}{dt} \to \dot{\vec{\gamma}}_0 \quad \text{strongly in } L^1$$

and

$$\vec{\gamma}_0 \text{ is a geodesic with } L(\vec{\gamma}_0) = \beta_0$$

The Proof of Struwe Monotonicity Trick - page 1

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Replace the original **pseudo-gradient** X_τ for E^τ by X_τ^σ

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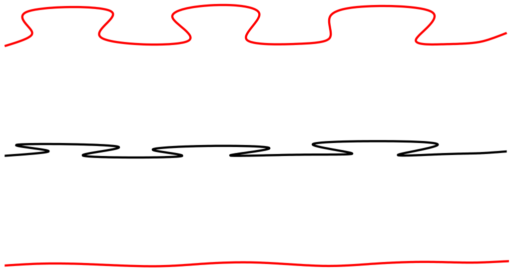
moreover

$$\xi \quad \text{is conformal} \quad \text{and} \quad \mathbf{v} \quad \text{is a } \textbf{stationary} \text{ varifold}$$

Varifold Convergence to Integer Rectifiable Varifold



$N = 3$

Parametrized Integer Rectifiable Stationary Varifolds

Let S be a Riemann Surface.

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Compare with the **harmonic map** equation

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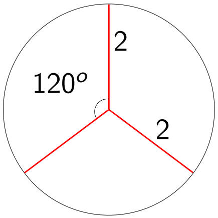
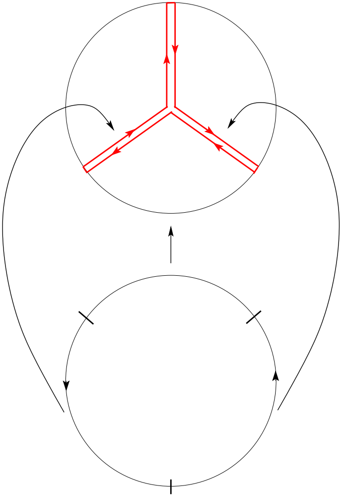
Theorem [R.2016] If $N \equiv N_0$, if (S, Φ, N) is a **Param. Int. Rect. Stat. Var.** then Φ is **harmonic**. □

Classical Integer Rectifiable Stationary Varifold



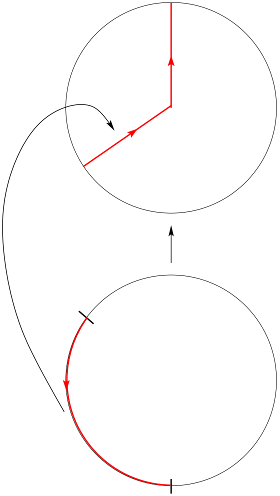
classical stationary varifolds

Parametrized Integer Rectifiable Stationary Varifolds



parametrized varifold

Parametrized Integer Rectifiable Varifolds



This is **not** a parametrized integer rectifiable stationary varifold

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Theorem[Allard, 1972] The space of integer rectifiable stationary varifold is weakly sequentially closed in the space of varifolds.

Theorem[Pigati, R. 2017] A parametrized integer rectifiable stationary varifold has **everywhere** integer multiplicity and a plane as tangent cone away from a dimension 0 set.

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Conclusion so far

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If $\beta(0) < \frac{8\pi}{\|\,\mathbb{I}\,\|_\infty}$ then (S, Φ, N) is a smooth minimal embedding.

Minimal Surface Genealogy in S^3 [R. 2017]

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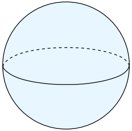

$W_1(S^3) = 4\pi$

Geodesic spheres $\simeq S^3$

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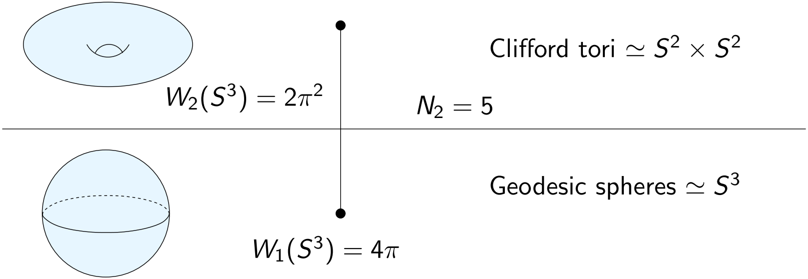

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